Exercises

1. Given two natural numbers $a, b \in \mathbb{N}$, prove that there is a natural number $m \in \mathbb{N}$ such that $m \cdot a > b$.

2. Let $a \in \mathbb{N}$. If the set X has the following property: $a \in X$ and $n \in X \Rightarrow n+1 \in X$. Then X contains all natural numbers greater than or equal to a.

3. A number $a \in \mathbb{N}$ is called **predecessor** of $b \in \mathbb{N}$ if $a < b$ and there is no $c \in \mathbb{N}$ such that $a < c < b$. Prove that every number, except 1, has a predecessor.

4. Show the following using induction:

   a. $2(1+2+\ldots+n) = n(n+1)$;

   b. $1+3+5+\ldots+(2n+1) = (n+1)^2$;

   c. $n \geq 4 \Rightarrow n! > 2^n$.

5. Using strong induction show that the decomposition of any number in prime factors is unique.

6. Let X be a finite set with n elements. Use induction to show that the set of all functions $f : X \to X$ has exactly $n!$ elements.

7. Let X be a finite set. Show that a function $f : X \to X$ is injective $\iff$ is surjective.

8. Give an example of a surjective function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, the set $f^{-1}(n)$ is infinite.

9. Show that the power set $\mathcal{P}(A)$ of a set A with n elements has 2^n elements.

10. Show that if A is countably infinite then $\mathcal{P}(A)$ is uncountable.

11. Let $f : X \to X$ be injective but not surjective. If $x \in X - f(X)$, show that $x, f(x), f(f(x)), \ldots$ are pairwise distinct.

12. Let X be an infinite set e Y a finite set. Show that there is a surjective function $f : X \to Y$ and an injective function $g : Y \to X$.

13. Find subsets $X_i \subseteq \mathbb{N}$ and a decomposition

$$\mathbb{N} = X_1 \cup X_2 \cup \ldots \cup X_i \cup \ldots,$$

    such that X_i are infinite sets and pairwise disjoints.

14. Let $X \subseteq \mathbb{N}$ be infinite. Show that there is a unique increasing bijection $f : \mathbb{N} \to X$.

15. A sequence of natural numbers $\{a_1, a_2, a_3, \ldots\}$ is called increasing if $a_i < a_{i+1}$. Show that the set of all increasing sequences of natural numbers is uncountable.

16. (Cantor-Bernstein-Schroder theorem) Given sets A and B, let $f : A \to B$ and $g : B \to A$ be injective functions. Show that there is a bijection $h : A \to B$.

17. Given a sequence of sets $A_1, A_2, A_3, \ldots$, we define the *limit superior* as the set

$$\limsup A_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{i=n}^{\infty} A_i \right).$$

Similarly, the *limit inferior* is the set

$$\liminf A_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{i=n}^{\infty} A_i \right).$$

a. Show that $\limsup A_n$ is the set of elements that belong to A_i for infinitely many values of i. Similarly, show that $\liminf A_n$ is the set of elements that belong to A_i for every value of i, except possibly, for a finite number of values of i.

b. Conclude that $\liminf A_n \subseteq \limsup A_n$.

c. Show that if $A_n \subseteq A_{n+1}$ for every n then $\liminf A_n = \limsup A_n = \bigcup_{n=1}^{\infty} A_n$.

d. Show that if $A_{n+1} \subseteq A_n$ for every n then $\liminf A_n = \limsup A_n = \bigcap_{n=1}^{\infty} A_n$.

e. Give an example of sequence such that $\liminf A_n \neq \limsup A_n$.